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NON-LINEAR NON-PLANAR RESONANT OSCILLATIONS IN FIXED-FREE BEAMS WITH SUPPORT ASYMMETRY

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Abstract—The order-three, integro differential, non-linear equations of motion for an inextensional beam, derived by the authors in a previous publication, are analyzed to investigate non-linear resonant coupling effects between the non-planar free oscillation modes of a fixed-free beam with asymmetric support conditions. The transition curves that separate non-linear resonant and non-resonant types of motions for the beam, and the main characteristics of the non-linear motions, are determined analytically.

INTRODUCTION

Non-linear oscillatory behavior of dynamical systems has frequently been the subject of investigations in the literature. Its interest stems mainly from the fact that the non-linearities present in the differential equations of motion may give rise to phenomena that are not disclosed by a linear approximation to these equations. Nayfeh[1] illustrates the analysis of non-linear motions using perturbation techniques for a large spectrum of systems governed by both ordinary and partial differential equations. Rosenberg[2] presents a very good early review of the subject of non-linear oscillations.

Of special interest in the theory of non-linear unforced oscillations is a resonance phenomenon due to non-linear coupling between two or more modes of the system with an exchange of energy between such modes. In the presence of such non-linear resonance, the amplitudes of one or more of the modes of the system may grow to very large values when compared to the norm of the non-zero initial conditions in that mode, regardless of how small those initial conditions are. The occurrence of this phenomenon in dynamical systems, such as satellites[3–5], ships at sea[6], and spring-mass systems[7–9], are a few examples of several investigations having the objective of predicting and controlling it. Agrawall and Evan-Iwanowski[10] analyzed a gyroscopic system for non-linear resonances, and presented an explanation of the phenomenon in terms of the virtual work done by the perturbing forces.

In this paper, non-linear resonance between the in-plane and out-of-plane free oscillation modes of a beam is investigated. Experiments have demonstrated that under certain conditions the oscillations tend to depart from the plane of the initial displacement even when those oscillations are "nominally" started along a principal plane of the beam[11].

Considerable attention has been given, in the literature, to the problem of non-linear motions of beams. The great bulk of the literature is, however, directed to the investigation of planar motions[12–29]. Reviews of the subject have been recently presented by Evan–Iwanowski[30,31]. Non-planar motions of beams has received relatively little attention, as exemplified in[32–36]. Most of the studies are based either on differential equations valid for systems in which torsional effects are neglected, or on equations obtained by linearization of the beam's curvature.

In the following, the non-planar motion of a fixed-free beam able to experience flexure about two principal axes, and twisting, is analyzed. The effects of torsional stiffness and support flexibility on the free non-linear motion of the beam are investigated in detail. The conditions under which motions nominally started along a principal plane of the beam leave that plane are determined. The subsequent motions are also investigated.

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EQUATIONS OF MOTION AND BOUNDARY CONDITIONS FOR THE FIXED-FREE BEAM

In previous work by the authors, the governing differential equations of motion suitable for analyzing non-linear, non-planar motions of an inextensional beam were developed[37,38]. The beam to be analyzed is assumed to be initially straight and untwisted, and to oscillate without appreciable [$0(\epsilon^n)$ with $n > 3$] damping, shear, warping, or extensional deformation, where ϵ is a small parameter associated with the initial amplitude of the oscillations, to be specified later.

It will also be assumed here that the effect of neglecting the principal distributed mass moments of inertia of the beam,† j_ξ, j_η, and j_ζ, is small. Deletion of the torsional mass moment of inertia j_ξ implies that the torsional frequency of the beam is much larger than both of its flexural frequencies. To illustrate the range of applicability of this assumption, we note that the flexural, and the torsional frequencies of a rectangular beam, ω_f and ω_t, shown in Fig. 1, are

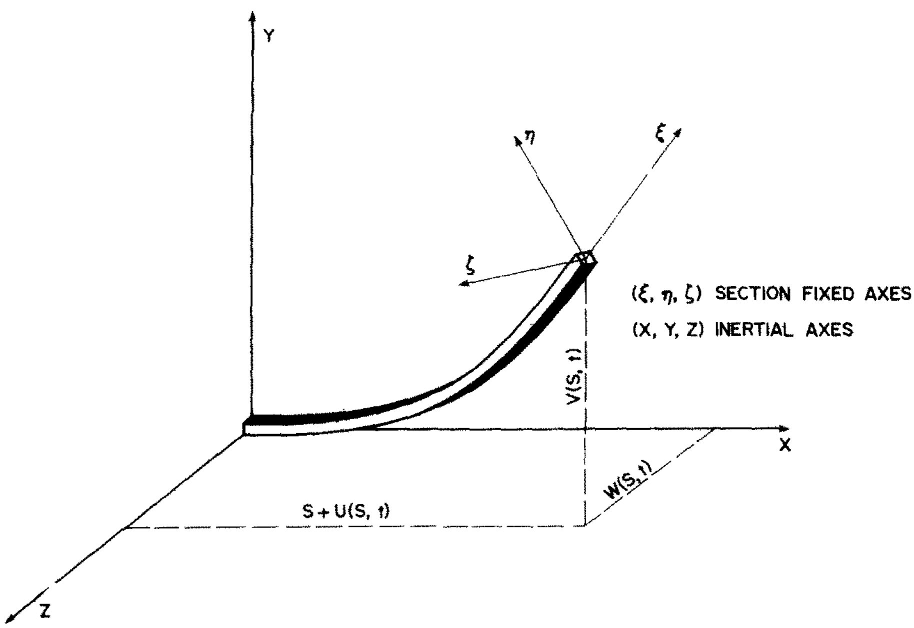

Fig. 1. The fixed-tree beam and the elastic displacement components of the centroid of the cross section at location s.

given by $\omega_f = 1.875^2[EI/(mL^4)]^{1/2}$ and $\omega_t = \pi[GK/j_\xi]^{1/2}/(2L)$, where m is the beam's mass per unit length, G is the shear modulus, I is the cross sectional area moment about either flexure axis, and K is the torsional stiffness as given in Ref. [39, p. 194]. From the expressions for ω_f and ω_t, it may be concluded that ω_t for a rectangular beam with a cross section aspect ratio smaller than three, is greater than ten times ω_f, when the length L of the beam is greater than twenty times the larger of its cross sectional dimensions. The above assumptions disallow consideration of resonance between torsional and lateral oscillation modes.

One objective of this paper is to determine the influence of support asymmetry on the non-linear behavior of the lateral modes of oscillation of the beam. As it will be shown, the influence of support asymmetry, and terms arising from non-linear expressions for the curvature, may play a significant role in the non-linear resonant motion of the beam. To this end, a spring-like support is considered. For simplicity, it is assumed that the base is rigid in the direction of the w-deflection (see Fig. 1). The relationship between the spatial derivatives of the transverse displacement $v(s, t)$, at the base, is written as‡

$$v'(s = 0, t) = \delta v''(s = 0, t) \tag{1a}$$

†The section fixed, principal inertia axes of the beam are shown in Fig. 1. The nomenclature used here is the same adopted in Refs. [37, 38].

‡Primes denote differentiation with respect to arc-length s, normalized by the length L of the beam. The two elastic displacements, $v(s, t)$ and $w(s, t)$, are normalized by the beam's length, and the time t by $[D_\eta/(mL^4)]$

where δ is a constant. The other boundary conditions for the system shown in Fig. 1 are

$$v(0, t) = w(0, t) = w'(0, t) = 0 \tag{1b}$$

$$v''(1, t) = w''(1, t) = v'''(1, t) = w'''(1, t) = 0. \tag{1c}$$

By defining $\beta_y = D_\zeta/D_\eta$ and $\beta_\gamma = D_\xi/D_\eta$ (where D_ξ, D_η and D_ζ represent, respectively, the torsional and principal flexural stiffnesses of the beam), and by neglecting damping, the order-three non-linear integro differential equations of motion for the free oscillations of the beam under consideration are given as[37,38].

$$\Big\{(1-\beta_y)\Big[w''\int_1^s v''w''\,ds - \beta_r\Big(w''\int_0^s\int_1^s v''w''\,ds\,ds\Big)' - w'''\int_0^s v''w'\,ds\Big] - \beta_y[v''' + v'(w'w'')' + v'(v'v'')'] - \frac{1}{2}v'\int_1^s\Big[\int_0^s (v'^2+w'^2)\,ds\Big]^{\cdot\cdot} ds\Big\}' = \ddot{v}. \tag{2a}$$

$$\Big\{(1-\beta_y)\Big[-v''\int_1^s v''w''\,ds - \beta_r(v''\int_0^s\int_1^s v''w''\,ds\,ds\Big)' + v'''\int_0^s w''v'\,ss\Big] - [w''' + w'(w'w'')' + w'(v'v'')'] - \frac{1}{2}w'\int_1^s\Big[\int_0^s (v'^2+w'^2)\,ds\Big]^{\cdot\cdot} ds\Big\}' = \ddot{w} \tag{2b}$$

where $\beta_r = (1-\beta_y)/\beta_\gamma$.

RESONANT NON-PLANAR MOTIONS

Equations (2a) and (2b) show that the non-linearities in the system, including torsional-flexural coupling, are cubic, which is not the case when the inertia j_ξ is included[40].

To determine an approximation for the maximum amplitude and the period of the non-planar resonant oscillations, two time scales, $t_0 = t$ and $t_2 = \epsilon^2 t$, are introduced[1]. The fast time scale t_0 is identified with the oscillation associated with the linearized counterpart of equations (2), while the slower time scale t_2 is associated with the slower changes in amplitudes and phases of the coupled non-linear motion.

Following Nayfeh[1], Crespo da Silva[8,9] and Glynn and Crespo da Silva[11], the coupled variables $v(s, t)$ and $w(s, t)$, and the flexure stiffness ratio β_y are expanded as

$$v(s, t; \epsilon) \sim \epsilon v_1(s, t_0, t_2) + \epsilon^3 v_3(s, t_0, t_2) \tag{3a}$$

$$w(s, t; \epsilon) \sim \epsilon w_1(s, t_0, t_2) + \epsilon^3 w_3(s, t_0, t_2) \tag{3b}$$

$$\beta_y \sim 1 + \Delta_0 + \epsilon^2\Delta_2. \tag{3c}$$

By substituting eqns (3) into eqns (2), the following linear, uncoupled, partial differential equations are obtained when the coefficients of the terms in equal powers of ϵ are equated to zero.

Order ϵ

$$d_0^2 v_1 + (1+\Delta_0)v_1'''' = 0 \tag{4a}$$

$$d_0^2 w_1 + w_1'''' = 0. \tag{4b}$$

Order ϵ^3

$$d_0^2 v_3 + (1+\Delta_0)v_3'''' = -2d_0d_2v_1 - \Delta_2 v_1'''' - \Big\{\Delta_0 w_1''\int_1^s v_1''w_1''\,ds + \frac{\Delta_0^2}{\beta_\gamma}\Big(w_1''\int_0^s\int_1^s v_1''w_1''\,ds\,ds\Big)' - \Delta_0 w_1'''\int_0^s v_1''w_1'\,ds + (1+\Delta_0)[v_1'(w_1'w_1'')' + v_1'(v_1'v_1'')] + \frac{1}{2}v_1'\int_1^s d_0^2\Big[\int_0^s (v_1'^2 + w_1'^2)\,ds\Big]ds\Big\}' \tag{4c}$$

$$d_0^2w_3 + w_3'''' = -2d_0d_2w_1 + \left\{\Delta_0 v_1''\int_0^s v_1''w_1''\,ds - \frac{\Delta_0^2}{\beta_\gamma}\left(v_1''\int_0^s\int_1^s v_1''w_1''\,ds\,ds\right)' - \Delta_0 v_1'''\int_0^s w_1''v_1'\,ds - w_1'(w_1'w_1'')' - w_1'(v_1'v_1'')' - \frac{1}{2}w_1'\int_1^s d_0^2\left[\int_0^s (v_1'^2 + w_1'^2)\,ds\right]ds\right\}'. \tag{4d}$$

From eqns (1) and (3), the boundary conditions for eqns (4) are obtained as

$$v_1(0,t) = w_1(0,t) = v_1'(0,t) - \delta v_1''(0,t) = w_1'(0,t) = 0 \tag{5a}$$

$$v_3(0,t) = w_3(0,t) = v_3'(0,t) - \delta v_3''(0,t) = w_3'(0,t) = 0 \tag{5b}$$

$$v_1''(1,t) = w_1''(1,t) = v_1'''(1,t) = w_1'''(1,t) = 0 \tag{5c}$$

$$v_3''(1,t) = w_3''(1,t) = v_3'''(1,t) = w_3'''(1,t) = 0. \tag{5d}$$

In this paper, only resonances between similar harmonics of the $v(s,t)$ and $w(s,t)$ modes will be explored. For this, the solutions to eqns (4) are approximated by one mode as

$$v_1(s,t_0,t_2) = \sqrt{(q(t_2))}F_v(s)\cos[r_1^2\sqrt{(1+\Delta_0)}t_0 + B_v(t_2)] = F_v(s)v_{1t}(t_0,t_2) \tag{6a}$$

$$w_1(s,t_0,t_2) = \sqrt{(p(t_2))}F(s)\cos[r^2t_0 + B_w(t_2)] = F(s)w_{1t}(t_0,t_2) \tag{6b}$$

where $F(s)$ and $F_v(s)$ are given as

$$F(s) = \cosh(rs) - K\sinh(rs) - \cos(rs) + K\sin(rs) \tag{7a}$$

$$F_v(s) = C[\cosh(r_1s) - K_1\sinh(r_1s) - \cos(r_1s) + K_2\sin(r_1s)]. \tag{7b}$$

The constants r and r_1 in eqns (6) and (7) are eigenvalues of the characteristic equations

$$1 + (\cosh r)\cos r = 0 \tag{8a}$$

$$1 + (\cosh r_1)\cos r_1 - r_1\delta[(\cosh r_1)\sin r_1 - (\sinh r_1)\cos r_1] = 0 \tag{8b}$$

and the constants K, K_1 and K_2 are given as

$$K = (\cosh r + \cos r)/(\sinh r + \sin r) \tag{9a}$$

$$K_1 = (\cosh r_1 + \cos r_1 - 2r_1\delta\sin r_1)/(\sinh r_1 + \sin r_1) \tag{9b}$$

$$K_2 = (\cosh r_1 + \cos r_1 + 2r_1\delta\sinh r_1)/(\sinh r_1 + \sin r_1). \tag{9c}$$

It can be verified that $\int_0^1 F^2(s)\,ds = 1$. The arbitrary constant C in eqn (7b) is chosen, for later convenience, such that $\int_0^1 F_v^2(s)\,ds = 1$.

An approximate solution to the non homogeneous partial differential equations (4c, d) can be obtained by transforming them to ordinary differential equations. Toward this end, two functions $v_{3t}(t_0,t_2)$ and $w_{3t}(t_0,t_2)$ are introduced as defined by eqns (10).

$$v_{3t}(t_0,t_2) = \int_0^1 v_3(s,t_0,t_2)F_v(s)\,ds \tag{10a}$$

$$w_{3t}(t_0,t_2) = \int_0^1 w_3(s,t_0,t_2)F(s)\,ds. \tag{10b}$$

By substituting eqns (6) into eqns (4c, d), multiplying the resulting equations by $F_v(s)$ and $F(s)$, respectively, and integrating both equations over the interval $s=0$ to $s=1$, the following ordinary differential equations are obtained for the variables $v_{3t}(t_0,t_2)$ and $w_{3t}(t_0,t_2)$

$$d_0^2v_{3t} + r_1^4(1+\Delta_0)v_{3t} = -2d_0d_2v_{1t} - \Delta_2r_1^4v_{1t} - \Delta_0(\alpha_{1v} + \Delta_0\alpha_{2v}/\beta_\gamma)v_{1t}w_{1t}^2 - (1+\Delta_0)(\alpha_{31v}w_{1t}^2 + \alpha_{32v}v_{1t}^2)v_{1t} - v_{1t}d_0^2(\alpha_{41v}v_{1t}^2 + \alpha_{42v}w_{1t}^2)/2 \tag{11a}$$

$$d_0^2 w_{3t} + r^4 w_{3t} = -2d_0 d_2 w_{1t} + \Delta_0(\alpha_{1w} - \Delta_0\alpha_{2w}/\beta_\gamma)v_{1t}^2 w_{1t} - (\alpha_{31w}v_{1t}^2 + \alpha_3 w_{1t}^2)w_{1t} - w_{1t}d_0^2(\alpha_4 w_{1t}^2 + \alpha_{4w}v_{1t}^2)/2. \quad (11b)$$

In eqns (11) the twelve "alpha coefficients" are constants defined in the Appendix. Equations (11) disclose that a resonance between the v and w motions occurs when $r_1^2\sqrt{(1+\Delta_0)} = r^2$.

To analyze the non-linear resonant motions of the beam, the above condition, and eqns (6), are substituted into eqns (11) to extract the solvability conditions from them. By introducing the quantities k_{1v}, k_{2v}, k_{2w}, Δ, μ_1 and μ_2, defined in the Appendix, and a new normalized slow time scale τ_2 as $\tau_2 = k_{1v}t_2$, the solvability conditions (namely, the conditions for v_{1t} and w_{1t} to be periodic), with dots now denoting differentiation with respect to τ_2, are

$$\dot{q} - pq \sin\psi = 0 \quad (12a)$$

$$\dot{p} + pq \sin\psi = 0 \quad (12b)$$

$$pq[\dot{\psi} + \mu_2 p - \mu_1 q + (q-p)\cos\psi + \Delta] = 0 \quad (12c)$$

where

$$\psi = 2(B_w - B_v).$$

From eqns (12) the following integrals of the motion are obtained,

$$q + p = \text{constant} = C_1 = q(0) + p(0) \quad (13a)$$

$$q[(\mu_2 - \cos\psi)(C_1 - q) + (\mu_2 - \mu_1)q/2 + \Delta] = \text{constant} = C_2 = q(0)\{[\mu_2 - \cos\psi(0)]p(0) + (\mu_2 - \mu_1)q(0)/2] + \Delta\}. \quad (13b)$$

Equations (13) may be used to eliminate the phase difference $\psi(\tau_2)$ and the variable $q(\tau_2)$ from eqn (12a) to obtain the following differential equation for $q(\tau_2)$

$$\dot{q} = \pm\{q^2(C_1 - q)^2 - [C_2 - (\mu_2 C_1 + \Delta)q + (\mu_1 + \mu_2)q^2/2]\}^{1/2} = \pm\sqrt{(f(q))}. \quad (14)$$

Of special interest is the non-linear resonant motion of the beam, given a finite displacement along one principal direction, say $w(s, t)$, and an infinitesimal initial amplitude along the $v(s, t)$ direction. For a resonant beam the non-planar component of the motion, $v(s, t)$, may grow to large values which are independent of the initial conditions in the $v(s, t)$ direction. To analyze such motion, let the perturbation parameter ϵ be equal to the initial amplitude of the $w(s, t)$ motion at the free end of the beam, $s = 1$. By noticing that $F(s = 1) = 2$, eqns (3), (6) and (13) yield,

$$\epsilon = \{[w^2 + (\partial w/\partial t)^2/r^4]^{1/2}/2\}_{s=1,\, t=0} \quad (15a)$$

$$p(0) = 1; \quad C_1 = 1 + q(0). \quad (15b)$$

In the limiting case $q(0) = 0$ (eqn 14), with $C_1 = 1$ and $C_2 = 0$, degenerates to

$$\dot{q} = \pm q\{[1 - (\mu_1 + \mu_2)^2/4] \cdot (q - 2q_{e1})(q - 2q_{e2})\}^{1/2} \quad (16)$$

where

$$q_{e1} = (1 - \mu_2 - \Delta)/(2 - \mu_1 - \mu_2) \quad (17a)$$

$$q_{e2} = (1 + \mu_2 + \Delta)/(2 + \mu_1 + \mu_2). \quad (17b)$$

Figure 2 shows the first harmonic ($\dot{q}$, q) trajectories for a beam with a compact rectangular cross section†, with $\delta = 0.5$ for the base, when the resonant condition previously obtained is satisfied. For infinitesimally small initial conditions $q(0)$, and for values of the parameter Δ in the range $-(1+\mu_2) < \Delta < 1 - \mu_2$, eqn (14) exhibits a maximum value for q, q_{max}, given as

†The torsional stiffness is taken from Ref. [41, p. 278] as $D_\xi = K_1 G a^3 b$, where K_1 is a constant tabulated in [41], and a and b are the dimensions of the cross section. The torsional to flexural stiffness ratio $\beta_\gamma = D_\xi/D_\eta$ is calculated as $\beta_\gamma \approx (1.687/(1+\mu))$. $\beta_y/(1+\beta_y) \approx 1.3\beta_y/(1+\beta_y)$, for a Poison's ratio of $\mu = 0.3$.

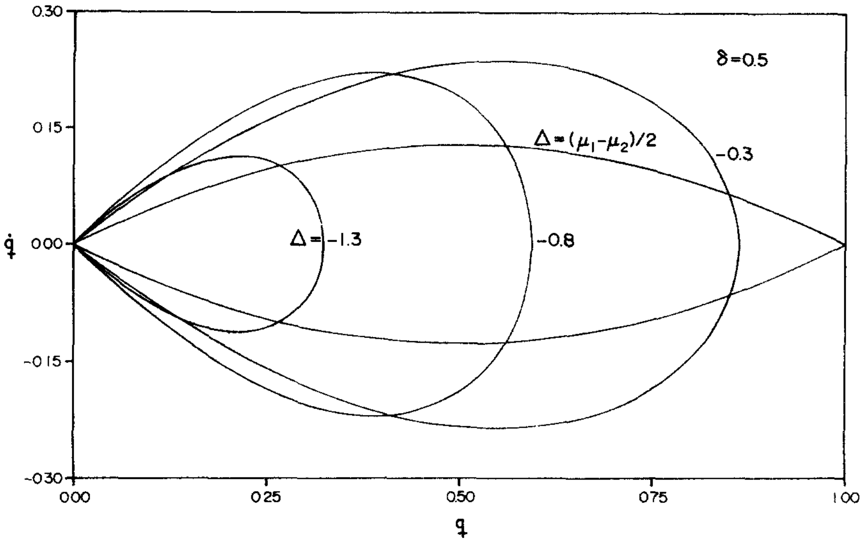

Fig. 2. First harmonic trajectories ($\dot{q}$, q) for several values of the parameter Δ, and for an asymmetric support with $\delta = 0.5$.

$q_{max} = \min(2q_{e1}, 2q_{e2}) + 0[q(0)]$ if $1-(\mu_1+\mu_2)^2/4 \geq 0$, or $q_{max} = \max(2q_{e1}, 2q_{e2}) + 0[q(0)]$ if $1-(\mu_1+\mu_2)^2/4 < 0$. The maximum amplitude of the resonant motion in the v-direction is practically independent of the infinitesimal initial conditions in that direction, and is given as $v_{max}(s, t) = \epsilon F_v(s)\surd(q_{max})$. For a base with $\delta = 0.5$, the maximum value of $F_v(s)$ for the first harmonic is $F_v(s = 1) \approx 1.82$.

The values $\Delta = -(1+\mu_2)$ and $\Delta = 1-\mu_2$ correspond to the transition curves between non-linear resonant and non-resonant motions, since in these cases $q_{max} = 0$ $[q(0)]$.

From eqn (3c), and the expressions for the parameters Δ, μ_1 and μ_2, given in the Appendix, the transition curves in the $(\beta_y - \epsilon)$ parameter space are obtained as

$$\beta_y = (1+\Delta_0)[1-\epsilon^2 k_{1v}(\mu_2 \pm 1)/r^2] + o(\epsilon^3). \tag{18}$$

Figure 3 shows the transition curves, for first harmonic interactions, given by eqn (18), and the region of non-linear resonance for the non-linearly coupled $v(s, t)$ and $w(s, t)$ motions of the beam. The resonant region on the left of that figure corresponds to a symmetric base

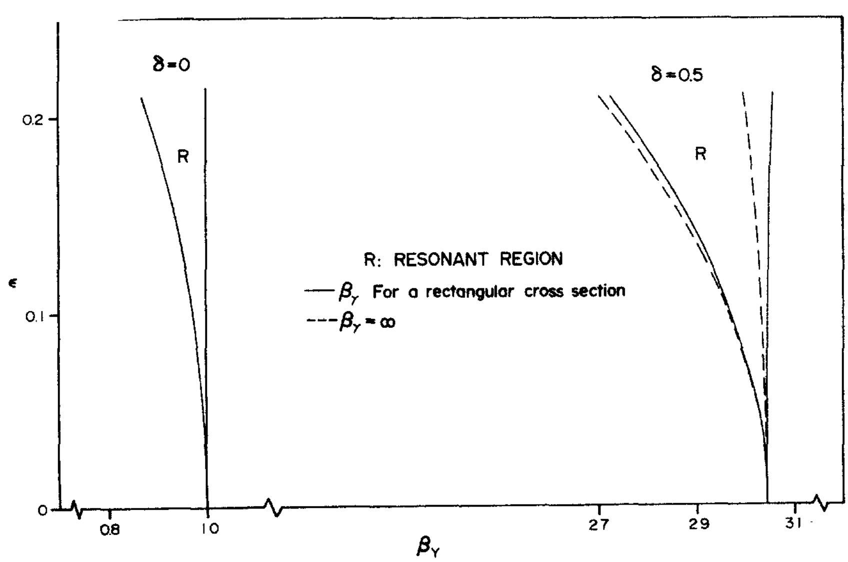

Fig. 3. Regions of resonance in the (ϵ, β_y) space for the non-linearly coupled transverse oscillations $v(s, t)$ and $w(s, t)$.

($\delta = 0$), while the region on the right corresponds to an asymmetric base with $\delta = 0.5$. To illustrate the effect of increasing the twisting stiffness ratio β_γ, the transition curves corresponding to $\beta_\gamma = \infty$ (i.e. a torsionally rigid beam) are also shown in Fig. 3. Note that for $\beta_\gamma = \infty$ the second non-linear term on the left hand sides of eqns (2a) and (2b) vanishes.

The period of the slow oscillations of the resonant v-motion may be obtained from eqn (14) in terms of an elliptic integral.

As a verification of the analytical results obtained in this paper, an approximate numerical solution to eqns (2) was sought. Toward this end, eqns (2) were transformed to ordinary differential equations by a Galerkin procedure with $v(s, t) \approx F_v(s)v_t(t)$ and $w(s, t) \approx F(s)w_t(t)$. The differential equations for $v_t(t)$ and $w_t(t)$ are given as

$$\ddot{v}_t(1 + \alpha_{41v}v_t^2) + \alpha_{42v}v_tw_t\ddot{w}_t = -r_1^4\beta_yv_t - v_t(\alpha_{41v}\dot{v}_t^2 + \alpha_{42v}\dot{w}_t^2) - \beta_yv_t(\alpha_{32v}v_t^2 + \alpha_{31v}w_t^2) + (1 - \beta_y)(\alpha_{1v} - \beta_r\alpha_{2v})v_tw_t^2 \quad (19a)$$

$$\ddot{w}_t(1 + \alpha_4w_t^2) + \alpha_{4w}v_tw_t\ddot{v}_t = -r^4w_t - w_t(\alpha_{4w}\dot{v}_t^2 + \alpha_4\dot{w}_t^2) - w_t(\alpha_{31w}v_t^2 + \alpha_3w_t^2) - (1 - \beta_y)(\alpha_{1w} + \beta_r\alpha_{2w})w_tv_t^2. \quad (19b)$$

Numerical values for all the alpha coefficients are given in the Appendix. Equations (19) were integrated numerically with the initial conditions $w_t(0) = \epsilon$, $v_t(0) = \epsilon/100$ and $\epsilon/1000$, and $\dot{v}_t(0) = \dot{w}_t(0) = 0$.

The maximum amplitude of the envelope of the slow v_t-oscillations obtained from the numerical integration of equations (19) was compared to the value obtained by the perturbation analysis. The results of this comparison are shown in Figs. 4 and 5 for several values of ϵ, as given by eqn (15a), for a beam with $\beta_\gamma = 1.3\beta_y/(1 + \beta_y)$. Also shown are the results for $\beta_\gamma = \infty$. The heavy lines in those figures represent the result of the analytical calculations, while the points indicated by a circle represent the results of the numerical integration of eqns (19). The results illustrated by Fig. 5, for the case of a symmetric base, are practically independent of the values of the torsional stiffness ratio, β_γ, under consideration. Figure 6 depicts the result of the numerical integration of eqns (19) for the resonant motions of the beam when it is supported by

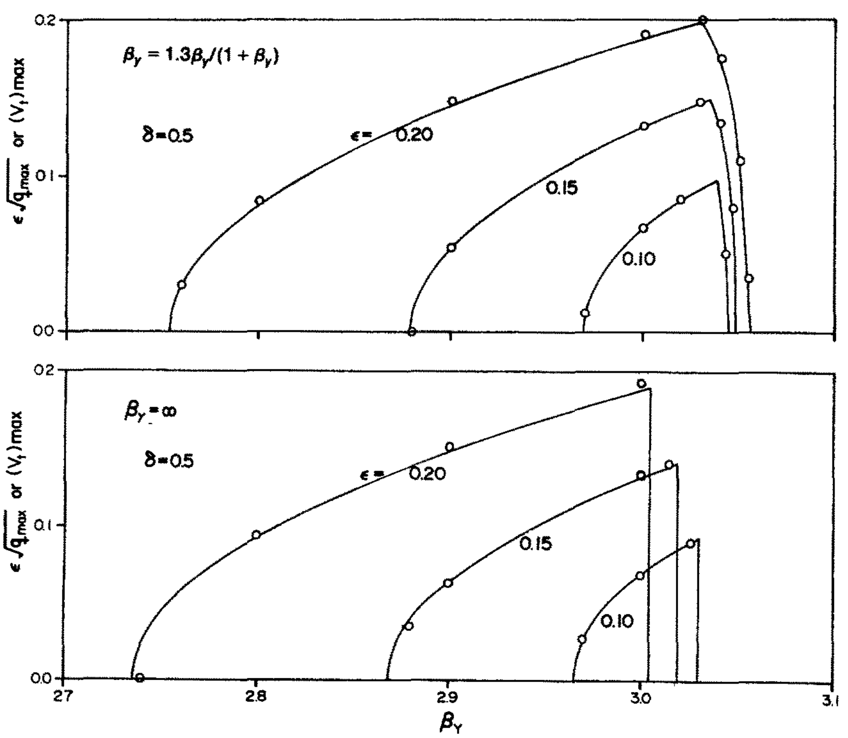

Fig. 4. Maximum resonant amplitude for the temporal variation of the first harmonic v-mode (asymmetric support).

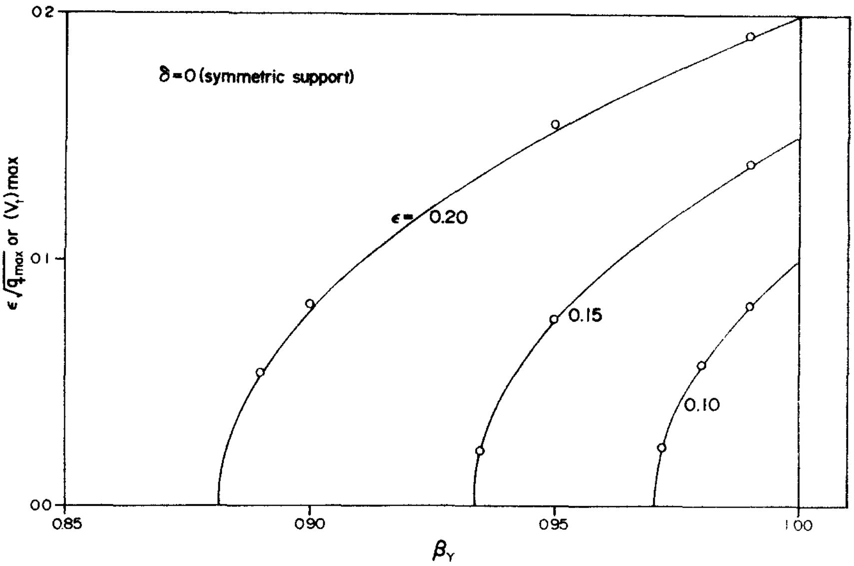

Fig. 5. Maximum resonant amplitude for the temporal variation of the first harmonic v-mode (symmetric support).

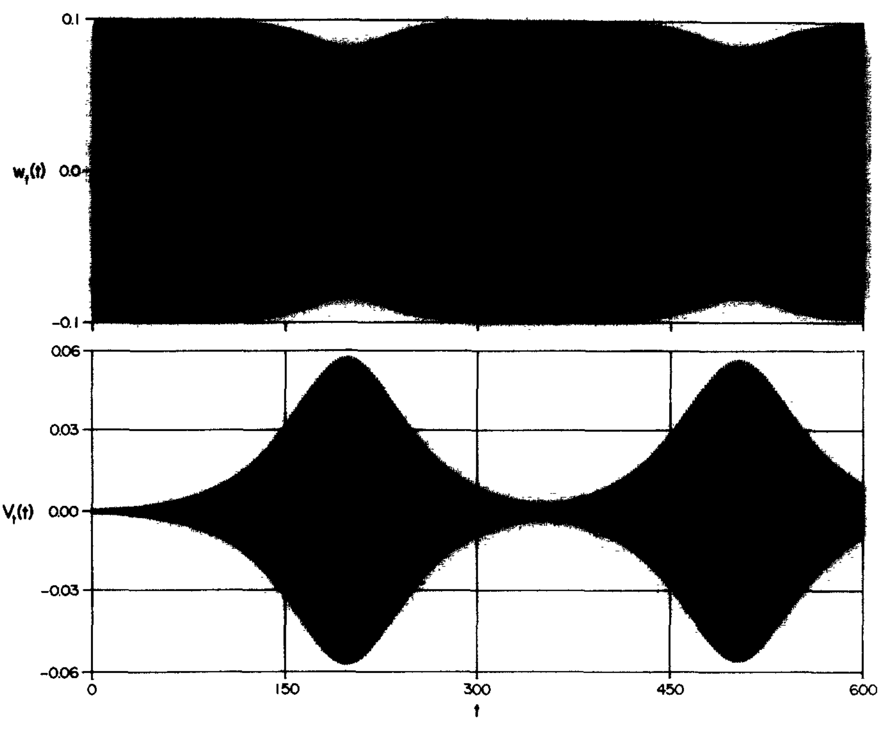

Fig. 6. Resonant motions $w_t(t)$ and $v_t(t)$ for $\beta_y = 0.98$ and $\epsilon = 0.1$ (symmetric support).

a symmetric base (i.e. $\delta = 0$), and for $\beta_y = 0.98$ and $\epsilon = 0.1$. Figure 7 illustrates the motion as viewed along the equilibrium x-axis of Fig. 1, for two values of β_y and for $\epsilon = 0.15$. The trajectories shown illustrate the history of the motion from zero time through one slow period of the oscillation.

SUMMARY AND CONCLUDING REMARKS

In this paper the effect of support asymmetry on the free, non-linear non-planar resonant motion of a fixed-free beam able to experience torsion and flexure is treated analytically. To the authors' knowledge, this is the first attempt to deal with the problem of non-planar free oscillations of a beam for which the flexural stiffness ratio is not near unity.

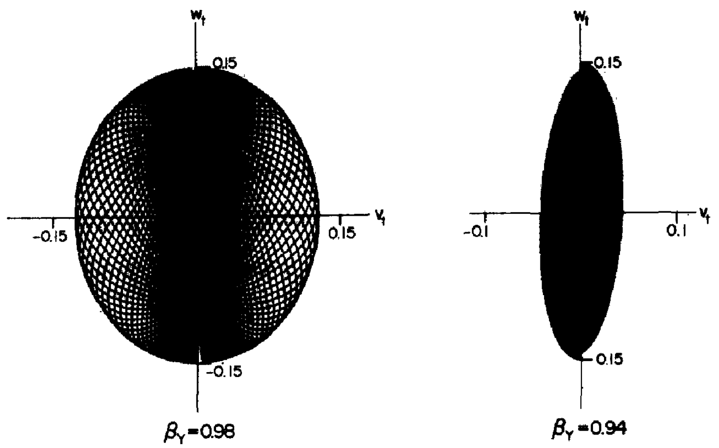

Fig. 7. $w_t(t)$ vs $v_t(t)$ trajectories for two values of β_y and $\epsilon = 0.15$.

The equations of motion used to analyze the behavior of the beam have been previously developed by the authors to retain the non-linear torsional-flexural coupling effects in a symmetric manner. Two constants of the motion were identified in the analysis, which enabled a single non-linear differential equation to completely represent the slow time variation of the out-of-plane or the in-plane oscillation amplitudes. The transition curves separating non-linear resonant and non-resonant behavior, in the flexural stiffness ratio vs the initial oscillation amplitude parameter space, were determined for first harmonic interactions.

The analytical results have been compared with results obtained by numerical integration of an approximate set of order-three differential equations of motion.

It was found that the torsional-flexural coupling is of consequence only when support conditions are asymmetric, and becomes increasingly significant, for a given flexural stiffness ratio, as the torsional rigidity is decreased. When the torsional to flexural stiffness ratio, β_γ, is increased beyond that for a rectangular cross section, the transition curves shown in Fig. 3 are shifted to the left in that figure. The opposite is true as the value of β_γ is reduced. It was also determined that even when the torsional coupling effect is neglected (i.e. $\beta_\gamma = \infty$) other non-linear terms multiplied by $(1-\beta_y)$ in eqns (2) could not be disregarded.

It appears that the non-linear coupling terms mentioned above may play a significant role in the motions when one examines interactions between different harmonics even when the support condition is symmetric.

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APPENDIX

The expressions for the twelve "alpha constants" introduced through eqns (11), and the constants k_{1v}, k_{2v}, k_{2w}, μ_1, μ_2 and Δ are given below. The values indicated were obtained numerically. They correspond to the first harmonic of the v and w motions for an asymmetric base with $\delta \mp 0.5$. The first value indicated for each coefficient corresponds to $\beta_\gamma = 1.3\beta_y/(1+\beta_y)$ and the second value, indicated in parenthesis, corresponds to $\beta_\gamma = \infty$.

$$\alpha_{1v} = \int_0^1 F_v(s)\left[F''(s)\int_1^s F_v''(s)F''(s)\,\mathrm{d}s - F'''(s)\int_0^s F_v''(s)F'(s)\,\mathrm{d}s\right]'\mathrm{d}s \approx -2.113\ (-2.113) \tag{A1}$$

$$\alpha_{2v} = \int_0^1 F_v(s)\left[F''(s)\int_0^s\int_1^s F''(s)F_v''(s)\,\mathrm{d}s\,\mathrm{d}s\right]''\mathrm{d}s \approx -1.742(-1.742) \tag{A2}$$

$$\alpha_{31v} = \int_0^1 F_v(s)\{F_v'(s)[F''(s)F'(s)]'\}'\,\mathrm{d}s \approx 14.23\ (14.23) \tag{A3}$$

$$\alpha_{32v} = \int_0^1 F_v(s)\{F_v'(s)[F_v'(s)F_v''(s)]'\}'\,\mathrm{d}s \approx 9.96\ (9.96) \tag{A4}$$

$$\alpha_3 = \int_0^1 F(s)\{[F'(s)F''(s)]'F'(s)\}'\,\mathrm{d}s \approx 40.44\ (40.44) \tag{A5}$$

$$\alpha_{41v} = \int_0^1 F_v(s)\left[F_v'(s)\int_1^s\int_0^s F_v'^2(s)\,\mathrm{d}s\,\mathrm{d}s\right]'\mathrm{d}s \approx 3.158\ (3.158) \tag{A6}$$

$$\alpha_{42v} = \int_0^1 F_v(s)\left[F_v'(s)\int_1^s\int_0^s F'^2(s)\,\mathrm{d}s\,\mathrm{d}s\right]'\mathrm{d}s \approx 3.76\ (3.76) \tag{A7}$$

$$\alpha_4 = \int_0^1 F(s)\left[F'(s)\int_1^s\int_0^s F'^2(s)\,\mathrm{d}s\,\mathrm{d}s\right]'\mathrm{d}s \approx 4.60\ (4.60) \tag{A8}$$

$$\alpha_{1w} = \int_0^1 F(s)\left[F_v''(s)\int_1^s F_v''(s)F''(s)\,\mathrm{d}s - F_v'''(s)\int_0^s F_v'(s)F''(s)\,\mathrm{d}s\right]'\mathrm{d}s \approx -12.12\,(-12.12) \tag{A9}$$

$$\alpha_{2w} = \int_0^1 F(s)\left[F_s''(s)\int_0^s\int_1^s F_v''(s)F''(s)\,\mathrm{d}s\,\mathrm{d}s\right]''\mathrm{d}s = \alpha_{2v} \tag{A10}$$

$$\alpha_{31w} = \int_0^1 F(s)\{F'(s)[F_v''(s)F_v'(s)]'\}'\,\mathrm{d}s = \alpha_{31v} \tag{A11}$$

$$\alpha_{4w} = \int_0^1 F(s)\left[F'(s)\int_1^s\int_0^s F_v'^2(s)\,\mathrm{d}s\,\mathrm{d}s\right]'\mathrm{d}s = \alpha_{42v} \tag{A12}$$

$$\Delta = \Delta_2 r^2/[(1+\Delta_0)k_{1v}] \tag{A13}$$

$$\mu_1 = k_{2v}/k_{1v} \approx 0.819\ (1.206) \tag{A14}$$

$$\mu_2 = k_{2w}/k_{1v} \approx 0.903\ (1.302). \tag{A15}$$

$$\begin{aligned} k_{1v} &= [2r^4\alpha_{42v} - \Delta_0(\alpha_{1v} + \Delta_0\alpha_{2v}/\beta_\gamma) - (1+\Delta_0)\alpha_{31v}]/(4r^2). \\ &= [2r^4\alpha_{4w} + \Delta_0(\alpha_{1w} - \Delta_0\alpha_{2w}/\beta_\gamma) - \alpha_{31w}]/(4r^2) \approx 4.36\ (3.84). \end{aligned} \tag{A16}$$

$$k_{2v} = [2r^4\alpha_{41v} - 3(1+\Delta_0)\alpha_{32v} - 2\Delta_0(\alpha_{1w} - \Delta_0\alpha_{2w}/\beta_\gamma) + 2\alpha_{31w}]/(4r^2) \approx 3.58\ (4.63). \tag{A17}$$

$$k_{2w} = [2r^4\alpha_4 - 3\alpha_3 + 2\Delta_0(\alpha_{1v} + \Delta_0\alpha_{2v}/\beta_\gamma) + 2(1+\Delta_0)\alpha_{31v}]/(4r^2) \approx 3.94\ (5.00). \tag{A18}$$

The eigenvalues r and r_1 of the characteristic equations (8a) and (8b) are $r \approx 1.875$ and $r_1 \approx 1.42$. For $\delta = 0$ (symmetric base), $k_{1v} = k_{2v} = k_{2w} \approx 5.20$.